Exponential and Logarithmic Functions

Finite Math

8 February 2017

Properties of Exponential Functions

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3. $a^x = b^x$ *for all* x *if and only if* $a = b$

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This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Growth and Decay Example

Example

In 2013, the estimated world population was 7.1 *billion people with a relative growth rate of* 1.1%.

(a) *Write a function modeling the world population* t *years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Now You Try It!

Example

The population of some countries has a relative growth rate of 3% per year. Suppose the population of such a country in 2012 is 6.6 million.

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Solution

(a) $P = 6.6e^{0.03t}$

(b) 7.67 *million;* 8.39 *million*

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Consider the function $f(x) = x^2$. If we run f backwards on the value 1, what x-value do we get?
Since $(1)^2 = 1$ and $(-1)^2 = 1$, we get *two* values when we run x^2 backward! So x^2 is not invertible.

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If we have a one-to-one function

$$y = f(x)$$

we can form the *inverse function* by switching x and y and solving for y:

$$x = f(y) \xrightarrow{\text{solve for } y} y = f^{-1}(x).$$

Logarithms

We will focus on one particular inverse function: the inverse of the function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

Definition (Logarithm)

The logarithm of base b is defined as the inverse of b^x. That is,

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Graphing a Logarithmic Function

Example

Sketch the graph of $f(x) = \log_2 x$.

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7. $\log_b M^p = p \log_b M$
8. $\log_b M = \log_b N$ *if and only if* $M = N$

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$$\log_b x = \frac{\ln x}{\ln b}$$

(See the textbook for a proof of this.)

Using Properties of Exponents and Logarithms

Example

Solve for x in the following equations:

(a) $7 = 2e^{0.2x}$

(b) $16 = 5^{3x}$

(c) $8000 = (x - 4)^3$

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